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Evolution time Klein–Gordon equation and derivation of its non-linear counterpart

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Abstract. Recently, relativistic quantum mechanical wave equations involving an extra scalar evolution parameter (analogous to proper time) have been increasingly studied in the literature. Since the physical meaning of this parameter is still very much a matter of discussion we aim to give it a clear physical meaning and then to proceed to a stochastic derivation of a non-linear evolution time Klein-Gordon equation with a view to representing possible vacuum dissipative effects.

1. Introduction

The use of an evolution parameter analogous to proper time in relativistic wave equations was first suggested by Fock [1] in 1937, and investigated further by Stückelberg [2] in 1941 and 1942. Stückelberg first considered a generalisation of classical relativistic mechanics to allow for backward-time worldlines through the parameter change

$$\tau_R = m_0 \tau \tag{1}$$

where we shall call τ (which is clearly a scalar parameter) the evolution time and τ_R is the usual proper time. His motive was to provide a model for pair production. Stückelberg extended his approach to relativistic quantum mechanics, but the only element to gain interest, with its reintroduction by Feynman [3], was the idea that the backward-time tracks represent antiparticles.

Apart from an article by Nambu [4] in 1950 on an evolution parameter and two parallel articles by Feynman [5] and Schwinger [6] in 1951 (the latter treating the evolution time as a mathematical auxiliary tool) very little development took place in the evolution time, until a publication in 1968 by Cooke [7]. In the same year, Pearle [8] reconsidered a generalisation of classical relativistic mechanics using the evolution time, and since then numerous articles on the use of evolution time have appeared. We mention in particular the work of Horwitz, Piron, Reuse, Soffer, and Rotbart [9], Fanchi [10], Collins [11, 12], Greenberger [13] and Hostler [14]. For an interesting review of the evolution time we refer the reader to a recent article by Kyprianidis [15].

In generalisations of classical relativistic mechanics the meaning of the evolution time τ is clear and is given by (1). It is just a scalar parameter proportional to the proper time τ_R. The transition to quantum mechanics allows states which, in general, represent a variable rest mass, so that relation (1) is not maintained and the meaning of τ in the wave equations becomes confused. Indeed, Hostler considered that this decouples τ from τ_R and allows τ to be treated as an independent parameter in the wavefunction $\psi(x, t, y, z, \tau)$. We shall show, however, that even in the quantum mechanical case relationship (1) lies at the heart of the meaning of τ and a clear relationship between τ and τ_R still exists.

The equation we consider is the evolution time Klein-Gordon equation:

$$\frac{\hbar^2}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\hbar^2\partial^2\psi}{\partial x^2}=-2\mathrm{i}\hbar\frac{\partial\psi}{\partial\tau} \tag{2}$$

which can be obtained from the relativistic energy relation

$$p^\mu p_\mu = m_0^2c^2 \tag{3}$$

by the following operator replacements:

$$p^\mu \rightarrow -\mathrm{i}\hbar\partial^\mu \qquad p_\mu \rightarrow -\mathrm{i}\hbar\partial_\mu \tag{4}$$

and

$$c^2m_0^2 \rightarrow 2\mathrm{i}\hbar\frac{\partial}{\partial\tau}. \tag{5}$$

We use the metric $g_{\mu\nu}$ with signature $(+, -, -, -)$. Refer to the appendix for a full list of the notation used.

The Hamiltonian operator $H = p^\mu p_\mu$ no longer represents the total energy, but rather the rest mass squared.

Plane wave solutions are

$$\psi = \exp\left[\frac{\mathrm{i}}{\hbar}\left(\omega t - \kappa x - \frac{m_0^2c^2\tau}{2}\right)\right] = \phi(x, t)\exp\left(-\frac{m_0^2c^2\tau}{2\hbar}\right). \tag{6}$$

These solutions represent particles of definite mass m_0, but since they are not normalisable, they are not strictly solutions. As usual, they may be regarded as useful idealisations. Instead, we must consider linear superpositions of (6) as solutions of (2). For example

$$\psi = \phi_0(x, t)\exp\left(-\frac{m_0^2c^2\tau}{2\hbar}\right) + \phi_1(x, t)\exp\left(-\frac{m_1^2c^2\tau}{2\hbar}\right). \tag{7}$$

Such superposed solutions represent a particle with a complicated oscillating rest mass, and it is this feature that prevents relation (1) from being maintained in the quantum mechanical case.

Apart from the physical meaning of τ we are also confronted with the question of the meaning of $\psi(x, y, z, t, \tau)$. The first point is that the use of the evolution time allows $\psi^*\psi$ to be interpreted as a positive-definite probability density:

$$\int_{-\infty}^{+\infty} \psi^*(x, y, z, t, \tau)\psi(x, y, z, t, \tau)\,\mathrm{d}x\,\mathrm{d}t = 1 \tag{8}$$

which is one important advantage and motivation for the use of evolution time in the Klein-Gordon equation. Based on (8) there have been two suggestions concerning the interpretation of ψ. Cooke [8] considered that (8) limits the existence of a particle in time and this led him to relate the meaning of ψ to an observation, rather than associating it with a particle. We shall adopt the alternative interpretation suggested by Fanchi and Collins [12], since it is analogous to the usual interpretation of ψ in non-relativistic quantum mechanics. They interpreted the wavefunction ψ as giving the probability of an event—an event corresponding to the position of a particle at a particular time. Incidentally, we mention that Fanchi and Collins [12] derived equation (2) by starting with (8) as their basic assumption.

The latter interpretation differs from the usual interpretation in that there is a probability distribution associated with time. Therefore, for each t there is some probability that a particle may or may not be found somewhere in space (hence Cooke's motivation for his interpretation). This is reasonable since particles such as mesons have finite lifetimes.

We may proceed to consider the physical meaning of the evolution time τ, and for this we need to introduce, briefly, a causal interpretation of the evolution time Klein-Gordon equation [16].

To begin, we decouple (2) into two real equations through the substitution $\psi = R\exp(iS/\hbar)$ where $R(x, y, z, t, \tau)$ and $S(x, y, z, t, \tau)$ are considered as real fields. This gives

$$\partial R^2/\partial\tau + \partial_\mu(R\partial^\mu S) = 0 \tag{9}$$

and

$$\frac{2\partial S}{\partial\tau} + \partial_\mu S\partial^\mu S - \frac{\hbar^2}{R}\partial_\mu\partial^\mu R = 0. \tag{10}$$

Equation (9) is to be regarded as a continuity equation expressing the conservation of $\psi^*\psi$ with respect to τ, and is clearly not a typical relativistic continuity equation. However, we only require that $\psi^*\psi = R^2$ be conserved with respect to τ for the probabilistic aspect of the theory to be retained. Thus we may define the expectation values of Hermitian operators, $\hat{O}$, representing physical observables as

$$\langle O\rangle = \int_{-\infty}^{+\infty} \psi^*\hat{O}\psi \, \mathrm{d}x\, \mathrm{d}t.$$

Equation (10) is the Hamilton-Jacobi equation with the extra term

$$QP = \frac{\hbar^2}{R}\partial_\mu\partial^\mu R \tag{11}$$

having the dimensions of rest mass squared, after division by c^2. It is the relativistic generalisation of Bohm's quantum potential, and we shall continue to refer to (11) as the quantum potential for convenience, if not with strict accuracy. We may recall that in the de Broglie [17] treatment of the usual Klein-Gordon equation, de Broglie defined a variable mass:

$$\mu_0^2 = (m_0^2 + QP/c)^{1/2} \tag{12}$$

which we shall require shortly, and note that in de Broglie's work m_0, which he called the bare mass, is a constant, whereas m_0 will generally be variable in our treatment.

A causal interpretation is achieved by defining the 4-momentum by analogy with classical relativistic Hamilton-Jacobi theory:

$$p^\mu = \partial^\mu S \qquad p_\mu = \partial_\mu S \tag{13}$$

with the additional definition,

$$m_0^2c^2 = -\tfrac{1}{2}\partial S/\partial\tau. \tag{14}$$

Also defining

$$p^{\mu} = \mathrm{d}x^{\mu}(\tau)/\mathrm{d}\tau \tag{15}$$

we can attribute a set of parallel world lines, $x^{\mu}(\tau)$, perpendicular to surfaces of constant S, with a particular wavefunction. The probabilistic aspect arises because we cannot measure the initial conditions and we cannot therefore tell on which worldline an event will occur.

We notice that the $x^{\mu}(\tau)$ are parametrised by τ, whereas we said earlier that in $\psi(x, y, z, t, \tau)$, τ is regarded as an independent parameter. This does not cause problems if we remember that, in the non-relativistic interpretation, t in $\phi(x, y, z, t)$ is regarded as an independent parameter, but the non-relativistic causal interpretation allows the definition of tracks $x(t)$, $y(t)$ and $z(t)$ parametrised by t. The use of τ in ψ can likewise be regarded as an independent parameter. Nor does the use of τ imply a five-dimensional space, any more than t in $\phi(x, y, z, t)$ implies a four-dimensional space in non-relativistic theory. τ, like t, is to be regarded as a scalar evolution parameter for four-dimensional spacetime, just as t is regarded as an evolution parameter in three-dimensional space.

Using (12)-(15) we can rewrite the Hamilton-Jacobi equation (10) as

$$c^2\mu_0^2 = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}x_{\mu}}{\mathrm{d}\tau}$$

so that

$$c^2\,\mathrm{d}\tau^2\mu_0^2 = \mathrm{d}x^{\mu}\,\mathrm{d}x_{\mu}. \tag{16}$$

Now $\mathrm{d}x^{\mu}\,\mathrm{d}x_{\mu}$ is an invariant distance and we may reparametrise using the proper time τ_R:

$$c^2\,\mathrm{d}\tau_R = \mathrm{d}x^{\mu}\,\mathrm{d}x_{\mu}. \tag{17}$$

Using (16) and (17) we get

$$\mu_0\tau = \tau_R \tag{18}$$

where the integration constant is chosen to be zero. This establishes a definite quantum mechanical relationship between τ and τ_R, contrary to Hostler's conclusion that, because of (1), the quantum mechanical treatment divorces τ from τ_R.

Consider further the expectation value of the rest mass given by

$$\langle m_0\rangle = \left(\frac{1}{c^2}\int \psi^*\hat{H}\psi\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}. \tag{19}$$

Since the mean contribution of the quantum potential to the de Broglie variable mass is zero, as may be seen from the relativistic counterpart of Eherenfest's theorem derived by Fanchi and Collins [12]:

$$\langle m_0^2c^2\rangle = \langle E^2\rangle + \langle \rho^2c^2\rangle \tag{20}$$

we may write

$$\langle\tau\rangle\langle m_0\rangle = \tau_R \tag{21}$$

and we see that Stückelberg's classical relativistic reparametrisation (1) holds also in the quantum mechanical treatment as an average relation.

In passing, we note that (15) and (18) immediately give the expression for the usual 4-velocity:

$$V^{\mu} = \frac{1}{\mu_0} p^{\mu} \tag{22}$$

tangential to the worldline $x^{\mu}(\tau)$.

We have introduced the evolution time Klein-Gordon equation at some length since it is, perhaps, not too familiar to the general reader. More importantly, the question of the physical meaning of the evolution time has been a matter of discussion and we felt is necessary to give at least one possible physical meaning of the evolution time before using it in our subsequent derivation. For an alternative very different interpretation of the evolution time we refer the reader to a recent article by Fanchi in 1986 [10]. We also felt it necessary to give some attention to the meaning of the wavefunction $\psi(x, y, z, t, \tau)$, which has not, perhaps, been given sufficient attention in the numerous publications on the subject. We accept, however, that we also have left untouched the many questions raised by the evolution time approach. A discussion of these difficult questions would take us too far from our second purpose, which is the derivation of a non-linear evolution time Klein-Gordon equation, representing possible vacuum dissipative effects.

2. Derivation of a non-linear evolution time Klein–Gordon equation representing possible vacuum dissipative effects

The derivation we give here is a relativistic generalisation of an earlier article [18] on the derivation of a non-linear Schrödinger equation. We recall briefly our motivation for this work.

It begins with the suggestion by Bohm and Vigier [19] of a subquantum aether, and the subsequent stochastic derivation of the Schrödinger equation by Nelson [20], for which the assumption of a subquantum aether is essential. On the mathematical side there has been considerable progress on the stochastic interpretation based on a subquantum aether [21], including relativistic generalisations to which we will shortly come, but there has been comparatively little study of the possible physical properties of such a subquantum aether. As a first step in this direction, and because effects such as the Unruh effect suggest vacuum dissipative effects, we proposed in the earlier article [18] to investigate the possibility that the subquantum aether possesses frictional dissipative effects. With an assumed frictional aether (of the covariant type suggested by Dirac [22]) and a generalisation of Nelson's stochastic derivation of the Schrödinger equation, we were led to a non-linear Schrödinger equation. Here again, we shall see that the assumed frictional aether leads to a non-linear equation; the non-linear evolution time Klein-Gordon equation. It is worth remarking that the use of an evolution time allows us to proceed in much the same way as for the non-relativistic case.

As stated in [18], severe limits are set on the possible properties of the subquantum aether by the experimental results at the quantum level, and for friction in particular, by the negative results of the Michelson-Morley experiment. Indeed, we used the experimental results for the Lamb shift to set an upper limit on the magnitude of our friction constant:

$$(\leqslant \ln\beta 1 + 4\times 10^{-13}).$$

We do not, therefore, expect to observe frictional aether effects at the quantum level, but we do expect such effects to manifest themselves at the cosmological level and we suggested a possible mechanism for the non-Doppler red shifts observed for some galaxies and how this would modify the Hubble law in closer accordance with observation [18]. We further mentioned in the same article that a relativistic generalisation is required for a proper treatment of the non-Doppler red shifts; in particular an extension to the Maxwell–Proca equations is required. As a first step in this direction we consider the simpler derivation of the non-linear evolution time Klein–Gordon equation.

In this paper, as in the first, we will not discuss solutions of the non-linear equations or the physical consequences. We have proposed and investigated various normalisable solutions, but we feel that further discussion and investigation is required, particularly with regard to the physical interpretation, before publication is possible. We therefore leave this important aspect for a later article.

Relativistic generalisations of Nelson's stochastic derivation of the Schrödinger equation have been considered by a number of authors; we mention Aron [23], de la Pena-Auerbach [24], Hakim [25], Lehr and Park [26], and from a more physical point of view, Vigier [27]. In particular, Hakim [25] has shown that the only value of the diffusion coefficient consistent with relativistic covariance is zero. Lehr and Park [26] have shown that this problem can be overcome by assuming time to be discrete. Further, to avoid the occurrence of spacelike stochastic jumps, Lehr and Park assumed that the stochastic jumps of the particle occur at the speed of light. To proceed we must also make similar assumptions. Thus, we assume that

(*a*) the evolution time is discrete and is characterised by an absolute minimum interval τ_c, sufficiently short to avoid inconsistency with observation;

(*b*) stochastic jumps occur at the velocity of light, or nearly so, since the presence of a frictional aether will affect this velocity very slightly (actually, negligibly, if we recall that the maximum value of our friction constant is $\beta \leqslant \ln(1+4\times 10^{-13})$).

To represent frictional effects in the non-relativistic case we proposed the use of a reduced time [18], first suggested by Levi-Civita [28] in 1896 and introduced into the quantum theory by Caldirola [29] and Kanai [30]. To represent friction in the subquantum aether in the relativistic case, we can generalise this idea by considering a reduced evolution time $^*\tau$, defined by

$$^*\tau = \chi(\tau)$$

so that,

$$d\tau = \eta(\tau)\, d^*\tau \qquad \eta(\tau) = \left(\frac{d\chi(\tau)}{d\tau}\right)^{-1} \tag{23}$$

where $^*\tau$ is the reduced evolution time. Henceforth, all modified quantities (not necessarily reduced) will be denoted by a * superscript.

Imposing the initial conditions $^*\tau = 0$ for $\tau = 0$ and $d^*\tau_0 = d\tau_0$ we get

$$\eta(\tau) = \exp\left(\int_0^{\tau} \gamma(\tau')\, d\tau'\right) \tag{24}$$

where $\gamma(\tau)$ has the dimensions mt^{-1} (mass × inverse time), and where we have approximated a sum (required by the discrete nature of τ) by an integral: this is a reasonable approximation since τ_c is necessarily very short. For simplicity, we assume simple friction for which γ is a positive constant, though we note that the derivation follows through for the more general $\gamma(\tau)$. With $\gamma=$ constant, $\eta(\tau)$ becomes

$$\eta(\tau)\,\mathrm{e}^{\gamma\tau} \qquad \text{so} \qquad \mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau=\mathrm{d}^*\tau \qquad \text{and} \qquad {}^*\tau=\frac{1-\mathrm{e}^{-\gamma\tau}}{\gamma} \tag{25}$$

which we note again are only approximations, though reasonably accurate ones, because of the discrete nature of τ.

With these definitions we can immediately generalise Nelson's stochastic definitions [20] for the forward and backward derivatives:

$$ {}^*D_+x^\mu(\tau)=\lim_{\Delta^*\tau\to\tau_c}\left\langle\frac{x^\mu(\tau+\Delta\tau)-x^\mu(\tau)}{\Delta^*\tau}\right\rangle=\left(\lim_{\Delta\tau\to\tau_c}\left\langle\frac{x^\mu(\tau+\Delta\tau)-x^\mu(\tau)}{\Delta\tau}\right\rangle\right)\mathrm{e}^{\gamma\tau}$$

$$ {}^*D_-x^\mu(\tau)=\lim_{\Delta^*\tau\to\tau_c}\left\langle\frac{x^\mu(\tau)-x^\mu(\tau-\Delta\tau)}{D^*\tau}\right\rangle=\left(\lim_{\Delta\tau\to\tau_c}\left\langle\frac{x^\mu(\tau)-x^\mu(\tau-\Delta\tau)}{\Delta\tau}\right\rangle\right)\mathrm{e}^{\gamma\tau}$$

so that

$$ {}^*D_+=\mathrm{e}^{\gamma\tau}D_+ \qquad \text{and} \qquad {}^*D_-=\mathrm{e}^{\gamma\tau}D_-. \tag{26}$$

The stochastic derivatives for a general function $F(x^\mu(\tau),\tau)$ of a stochastic process $x^\mu(\tau)$ with respect to the reduced evolution time are

$$\begin{aligned} {}^*D_+F&=\left(\frac{\partial}{\partial^*\tau}+{}^*p_+^\mu\,\partial_\mu+{}^*\nu\partial^\mu\partial_\mu\right)F\\ {}^*D_-F&=\left(\frac{\partial}{\partial^*\tau}+{}^*p_-^\mu\,\partial_\mu-{}^*\nu\partial^\mu\partial_\mu\right)F \end{aligned} \tag{27}$$

with ${}^*\nu$ the reduced evolution time diffusion coefficient.

We begin our derivation by defining the stochastic process $x^\mu(\tau)$ giving the coordinates of an event for a given τ and which incorporates the frictional effects of the subquantum aether:

$$\mathrm{d}x^\mu(\tau)={}^*p_+^\mu(x^\mu(\tau),\tau)\,\mathrm{d}^*\tau+\mathrm{d}W_+^\mu(\tau) \tag{28}$$

for the forward process and

$$\mathrm{d}x^\mu(\tau)={}^*p_-^\mu(x^\mu(\tau),\tau)\,\mathrm{d}^*\tau+\mathrm{d}W_-^\mu(\tau) \tag{29}$$

for the backward process. The $W(\tau)$ are Wiener processes satisfying

$$\begin{aligned} &\langle\mathrm{d}W_+^\mu\rangle=\langle\mathrm{d}W_-^\mu\rangle=0\\ &\langle\mathrm{d}W_+^\mu\,\mathrm{d}W_\nu^+\rangle=2\delta_\nu^\mu{}^*\nu\,\mathrm{d}^*\tau=2\delta_\nu^\mu\nu\,\mathrm{d}\tau \qquad \mathrm{d}\tau>0\\ &\langle\mathrm{d}W_-^\mu\,\mathrm{d}W_\nu^-\rangle=-2\delta_\nu^\mu{}^*\nu\,\mathrm{d}^*\tau=-2\delta_\nu^\mu\nu\,\mathrm{d}\tau \qquad \mathrm{d}\tau<0. \end{aligned} \tag{30}$$

Clearly, we have

$$ {}^*\nu=\nu\,\mathrm{e}^{\gamma\tau} \tag{31}$$

where ν represents the diffusion coefficient.

With these definitions, the forward and backward 4-momenta are given by

$$\begin{aligned} {}^*D_+x^\mu(\tau)&={}^*p_+^\mu=p_+^\mu\,\mathrm{e}^{\gamma\tau}\\ {}^*D_-x^\mu(\tau)&={}^*p_-^\mu=p_-^\mu\,\mathrm{e}^{\gamma\tau}. \end{aligned} \tag{32}$$

Remembering that τ is not the proper time we note that (32) gives the 4-momenta and not the 4-velocities.

The probability density $\rho(x, y, z, t, \tau)$ associated with a stochastic process $x^\mu(\tau)$ satisfies the generalised Fokker-Planck equations:

$$\begin{aligned} \partial\rho/\partial^*\tau &= -\partial_\mu(\rho^* p_+^\mu) + {}^*\nu\partial_\mu\partial^\mu\rho \equiv \partial\rho/\partial\tau = -\partial_\mu(\rho p_+^\mu) + \nu\partial_\mu\partial^\mu\rho \\ \partial\rho/\partial^*\tau &= -\partial_\mu(\rho^* p_-^\mu) - {}^*\nu\partial_\mu\partial^\mu\rho \equiv \partial\rho/\partial\tau = -\partial_\mu(\rho p_-^\mu) - \nu\partial_\mu\partial^\mu\rho. \end{aligned} \tag{33}$$

The average of these two equations gives our generalised continuity equation (9):

$$\partial\rho/\partial^*\tau + \partial_\mu(\rho^* p^\mu) = 0$$

or equivalently

$$\partial\rho/\partial\tau + \partial_\mu(\rho p^\mu) = 0 \tag{34}$$

where p^μ denotes the average drift 4-momentum, from which the average drift lines can be obtained. It is defined by

$$p^\mu = \tfrac{1}{2}(p_+^\mu + p_-^\mu) \qquad {}^*p^\mu = \tfrac{1}{2}({}^*p_+^\mu + {}^*p_-^\mu) \tag{35}$$

so that

$${}^*p^\mu = p^\mu\, e^{\gamma\tau}. \tag{36}$$

The osmotic 4-momentum is defined as

$$O^\mu = \tfrac{1}{2}(p_+^\mu - p_-^\mu) \tag{37}$$

with the modified form given by

$${}^*O^\mu = \tfrac{1}{2}({}^*p_+^\mu - {}^*p_-^\mu) \tag{38}$$

and so again

$${}^*O^\mu = O^\mu\, e^{\gamma\tau}. \tag{39}$$

The osmotic momentum is alternatively given by

$${}^*O^\mu = {}^*\nu\,\partial^\mu\rho/\rho \qquad O^\nu = \mu\,\partial^\mu\rho/\rho \qquad \rho = R^2(x, y, z, t, \tau). \tag{40}$$

The generalisation of Nelson's stochastic definition of acceleration [20] is

$${}^*f^\mu = \tfrac{1}{2}{}^*D_+{}^*D_-x^\mu(\tau) + \tfrac{1}{2}{}^*D_-{}^*D_+x^\mu(\tau) \tag{41}$$

where ${}^*f^\mu$ is a generalised force quantity, again because τ is not the proper time. Applying (41) to (32) and rearranging terms we get

$$\partial^*p^\mu/\partial\tau = {}^*f^\mu - ({}^*p^\nu\partial_\nu){}^*p^\mu + ({}^*O^\nu\partial_\nu){}^*O^\mu + {}^*\nu(\partial^\nu\partial_\nu)O^\mu. \tag{42}$$

Using (36) and (39), (42) becomes

$$\partial p^\mu/\partial\tau + \gamma p^\mu = f^\mu - (p^\nu\partial_\nu)p^\mu + (O^\nu\partial_\nu)O^\mu + \nu(\partial^\nu\partial_\nu)O^\mu \tag{43}$$

where γp^μ represents the frictional effect on the 4-momentum and

$${}^*f^\mu = f^\mu\, e^{2\gamma\tau} \tag{44}$$

(where f^μ is a generalised force quantity) since it is a second derivative with respect to τ. We shall consider this generalised force as being derived from a 4-potential $A^\mu \equiv (\phi, \boldsymbol{A})$:

$$f^\mu = (e/c)(\partial^\mu A^\nu - \partial^\nu A^\mu)p_\nu. \tag{45}$$

Assume the 4-momentum to be given by

$$\partial^\mu S = p^\mu + (e/c)A^\mu \tag{46}$$

with p^μ the generalised 4-momentum associated with the particle. Note

$$E = \partial S/\partial t \qquad \text{and} \qquad p_i = -\partial S/\partial x_i \qquad i = 1, 2, 3. \tag{47}$$

Substituting (46) into (45) and remembering that the curl of a gradient is zero we get for f^μ:

$$\begin{aligned} f^\mu &= (e/c)(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\nu S - (e/c)A_\nu) \\ &= -(e/c)(\partial^\nu A^\mu - \partial^\mu A^\nu)(\partial_\nu S - (e/c)A_\nu). \end{aligned} \tag{48}$$

Substituting (46) and (47) into (43), and using the relation

$$-p^\nu \partial_\nu p^\mu = (e/c)(\partial^\nu A^\mu - \partial^\mu A^\nu)(\partial_\nu S - (e/c)A_\nu) - \tfrac{1}{2}\partial^\mu(\partial_\nu S - (e/c)A_\nu)(\partial^\nu S - (e/c)A^\nu) \tag{49}$$

together with $A^\mu = \partial^\mu B$, where B is a scalar potential, we get, after simplification,

$$2\frac{\partial S}{\partial \tau} = -\left(\partial^\mu S - \frac{e}{c}A^\mu\right)\left(\partial_\mu S - \frac{e}{c}A_\mu\right) + \nu^2\left(\frac{\partial^\mu \partial_\mu R}{R}\right) - \gamma(S - B). \tag{50}$$

We notice that the curl term in (49) cancels the force term in (43) and note that

$$\partial A^\mu/\partial \tau = 0.$$

For the final step we substitute in (50) the diffusion coefficient ν†,

$$\nu = \hbar(1-\beta) \qquad \beta = \alpha\gamma$$

with $\beta \leqslant \ln(1 + 4\times 10^{-13})$ discussed in our earlier article [18], and use $\psi = R\exp(\mathrm{i}S/\hbar)$. We finally get the non-linear evolution time Klein-Gordon equation with an electromagnetic potential:

$$\begin{aligned} 2\mathrm{i}\hbar\frac{\partial \varphi}{\partial \tau} &= \left(-\mathrm{i}\hbar\partial^\mu - \frac{e}{c}A^\mu\right)\left(-\mathrm{i}\hbar\partial_\mu - \frac{e}{c}A_\mu\right)\psi + \beta\hbar^2\frac{\partial^\mu\partial_\mu(\psi^*\psi)^{1/2}}{(\psi^*\psi)^{1/2}}\psi \\ &\quad - \gamma[\mathrm{i}\hbar\ln(\psi/\psi^*)^{1/2} + B]\psi. \end{aligned} \tag{51}$$

3. Conclusion

We have introduced the use of an evolution time τ in relativistic wave equations, which has been considered by a number of authors, as mentioned in our introduction. We restricted ourselves only to a discussion of certain questions raised by this approach. In particular, we addressed the question of the physical meaning of τ and of the interpretation of $\psi(x, y, z, t, \tau)$.

We went on to extend an earlier investigation [18] concerning the possibility of a frictional subquantum aether. On the basis of this assumption we derived a non-linear evolution time Klein-Gordon equation, though we have left a discussion of the solutions and their physical interpretation for a later article. As a consequence of this work we hope to provide a physical mechanism for the non-Doppler red shifts, briefly discussed in our earlier article [18]. A proper treatment of the non-Doppler red shifts and the modifications to the Hubble law will require an application of our idea of a frictional subquantum aether to the Proca-Maxwell equations. We regard the present derivation, aside from its intrinsic interest, as a first step in this direction.

† For dimensional consistency we define a dimensionless constant $\beta = \alpha\gamma$, where α has dimensions mt^{-1}. α is not physically significant and may be chosen equal to unity.

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Notation

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$E = \frac{\partial S}{\partial t} \qquad p_x = -\frac{\partial S}{\partial x} \qquad p_y = -\frac{\partial S}{\partial y} \qquad p_z = -\frac{\partial S}{\partial z}$$

$$E \to -\mathrm{i}\hbar\frac{\partial}{\partial t} \qquad p_x \to \mathrm{i}\hbar\frac{\partial}{\partial x} \qquad p_y \to \mathrm{i}\hbar\frac{\partial}{\partial y} \qquad p_z \to \mathrm{i}\hbar\frac{\partial}{\partial z}$$

$$p^\mu = \partial^\mu S \qquad p_\mu = \partial_\mu S \qquad p^\mu = -\mathrm{i}\hbar\partial^\mu \qquad p_\mu = -\mathrm{i}\hbar\partial_\mu$$

$$p_\mu \equiv (E/c, -p_x, -p_y, -p_z) \qquad p^\mu = (E/c, p_x, p_y, p_z)$$

$$\partial_\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$$

$$\partial_\mu S \equiv \left(\frac{1}{c}\frac{\partial S}{\partial t}, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}\right) \qquad \partial^\mu S \equiv \left(\frac{1}{c}\frac{\partial S}{\partial t}, -\frac{\partial S}{\partial x}, -\frac{\partial S}{\partial y}, -\frac{\partial S}{\partial z}\right).$$

Electromagnetic potential (Gaussian system of units is used):

$$A_\mu = (\phi, -A_x, -A_y, -A_z) \qquad A^\mu = (\phi, A_x, A_y, A_z)$$

$$p_\mu \to p_\mu - (e/c)A_\mu \qquad p^\mu \to p^\mu - (e/c)A^\mu$$

$$\partial_\mu S = p_\mu + (e/c)A_\mu \qquad \partial^\mu S = p^\mu + (e/c)A^\mu.$$

Lorentz gauge:

$$\partial^\mu A_\mu = 0.$$

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